

THE MIXED PROBLEM FOR HARMONIC FUNCTIONS IN POLYHEDRA OF \mathbb{R}^3

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ABSTRACT. R. M. Brown's theorem on mixed Dirichlet and Neumann boundary conditions is extended in two ways for the special case of polyhedral domains. A (1) more general partition of the boundary into Dirichlet and Neumann sets is used on (2) manifold boundaries that are not locally given as the graphs of functions. Examples are constructed to illustrate necessity and other implications of the geometric hypotheses.

1. INTRODUCTION

In [Bro94] R. M. Brown initiated a study of the *mixed boundary value problem* for harmonic functions in *creased Lipschitz domains* Ω with data in the Lebesgue and Sobolev spaces $L^2(\partial\Omega)$ and $W^{1,2}(\partial\Omega)$ (with respect to surface measure ds) taken in the strong pointwise sense of nontangential convergence.

At the end of his article Brown poses a question concerning a certain topologic-geometric difficulty not included in his solution: Can the mixed problem be solved in the (infinite) pyramid of \mathbb{R}^3 , $|X_1| + |X_2| < X_3$, when Neumann and Dirichlet data are chosen to alternate on the faces? In this article we avoid the geometric difficulties of what can be called Lipschitz faces or facets and provide answers in the case of compact polyhedral domains of \mathbb{R}^3 . Some other recent approaches to the mixed problem for second order operators and systems in polyhedra can be found in [MR07] [MR06] [MR05] [MR04] [MR03] [MR02] and [Dau92].

Consider a compact polyhedron of \mathbb{R}^3 with the property that its interior Ω is connected. Ω will be termed a *compact polyhedral domain*. Suppose its boundary $\partial\Omega$ is a connected 2-manifold. Such a domain Ω need not be a Lipschitz domain. Partition the boundary of Ω into two disjoint sets N and D , for Neumann data and Dirichlet data respectively, so that the following is satisfied.

- (1.1) (i) N is the union of a number (possibly zero) of closed faces of $\partial\Omega$.
(ii) $D = \partial\Omega \setminus N$ is nonempty.
(iii) Whenever a face of N and a face of D share a 1-dimensional edge as boundary, the dihedral angle measured in Ω between the two faces is *less* than π .

The L^2 -polyhedral mixed problem for harmonic functions is

- (1.2) Given $f \in W^{1,2}(\partial\Omega)$ and $g \in L^2(N)$ show there exists a solution to $\Delta u = 0$ in Ω such that
(i) $u \rightarrow^{n.t.} f$ a.e. on D .
(ii) $\partial_\nu u \rightarrow^{n.t.} g$ a.e. on N .
(iii) $\nabla u^* \in L^2(\partial\Omega)$.

Here ∇u^* is the *nontangential maximal function* of the gradient of u . Generally for a function w defined in a domain G

$$w^*(P) = \sup_{X \in \Gamma(P)} |w(X)|, P \in \partial G.$$

For a choice of $\alpha > 0$ *nontangential approach regions* for each $P \in \partial G$ are defined by

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$$(1.3) \quad \Gamma(P) = \{X \in G : |X - P| < (1 + \alpha) \text{dist}(X, \partial G)\}$$

Varying the choice of α yields nontangential maximal functions with comparable $L^p(\partial G)$ norms $1 < p \leq \infty$ by an application of the Hardy-Littlewood maximal function. Therefore α is suppressed. In general when $w^* \in L^p(\partial G)$ is written it is understood that the nontangential maximal function is with respect to cones determined by the domain G . The *outer unit normal* vector to Ω (or a domain G) is denoted $\nu = \nu_P$ for *a.e.* $P \in \partial\Omega$ and the limit of (ii) is understood as

$$\lim_{\Gamma(P) \ni X \rightarrow P} \nu_P \cdot \nabla u(X) = g(P)$$

and similarly for (i).

A consequence of solving (1.2) is that the gradient of the solution has well defined nontangential limits at the boundary *a.e.*

In addition, as Brown points out, solving the mixed problem yields *extension operators* $W^{1,2}(D) \rightarrow W^{1,2}(\partial\Omega)$ by $f \mapsto u|_{\partial\Omega}$ where u is a solution to the mixed problem with $u|_D = f$. Consequently problem (1.2) cannot be solved for all $f \in W^{1,2}(D)$ when D and N are defined as on the boundary of the pyramid. For example, since the pyramid is Lipschitz at the origin so that Sobolev functions on its boundary project to Sobolev functions on the plane, solving (1.2) implies that a local $W^{1,2}$ function exists in \mathbb{R}^2 that is identically 1 in the first quadrant and identically zero in the third. Such a function necessarily restricts to a local $W^{\frac{1}{2},2}$ function on any straight line through the origin. But a step function is not locally in $W^{\frac{1}{2},2}(\mathbb{R})$. The boundary domain D (and its projection) do not satisfy the *segment property* commonly invoked to show the two Sobolev spaces $H_1(D)$ and $W^{1,2}(D)$ equal [Agm65] [GT83].

The admissible Sobolev functions on D must then be those that have extensions to $W^{1,2}(\partial\Omega)$. Or equivalently, the admissible Sobolev functions on D are the restrictions of $W^{1,2}(\partial\Omega)$ functions. We introduce the following *norm* on the *space of restrictions of $W^{1,2}(\partial\Omega)$ functions f to D*

$$\|f\|_D^2 = \inf_{\tilde{f}|_D = f} \int_{\partial\Omega} \tilde{f}^2 + |\nabla_t \tilde{f}|^2 ds$$

Here \tilde{f} denotes all $W^{1,2}(\partial\Omega)$ functions that restrict to f on D , and ∇_t denotes the tangential gradient. That this is a norm follows by arguments such as: Given $f \in W^{1,2}(\partial\Omega)$ and a real number a , the functions $a\tilde{f}$ form a subset of all extensions \tilde{af} of $(af)|_D$ so that $\|af\|_D \leq |a|\|f\|_D$, and thus likewise $\|f\|_D \leq |a|^{-1}\|af\|_D$ when $a \neq 0$.

This normed space is complete by using the standard completeness proof for Lebesgue spaces: Given a Cauchy sequence $\{f_j\}$ let j_k be such that $\|f_i - f_j\|_D < 2^{-k}$ for all $i, j \geq j_k$ and define $g_k = f_{j_{k+1}} - f_{j_k}$. Then there exists an extension \tilde{g}_k such that $\|\tilde{g}_k\|_{W^{1,2}(\partial\Omega)} < 2^{-k}$. Extensions of $f_{j_{n+1}}$ may then be defined by $\tilde{f}_{j_1} + \sum_{k=1}^n \tilde{g}_k$ Cauchy in $W^{1,2}(\partial\Omega)$. Completeness will follow. The Banach space of restrictions to D is undoubtedly the generally smaller Sobolev space $H_1(D)$ (e.g. [Fol95] p. 220), but this will not be pursued further.

A homogeneous Sobolev semi-norm on D is defined by

$$(1.4) \quad \|f\|_{D^o}^2 = \inf_{\tilde{f}|_D = f} \int_{\partial\Omega} |\nabla_t \tilde{f}|^2 ds$$

When $\partial\Omega$ is connected the following *scale invariant* theorem is established in the Section 2.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a compact polyhedral domain with connected 2-manifold boundary $\partial\Omega = D \cup N$ satisfying the conditions (1.1). Then given $f \in W^{1,2}(\partial\Omega)$ and $g \in L^2(N)$ there exists a unique solution u to the mixed problem (1.2). In addition there is a constant C independent of u such that*

$$\int_{\partial\Omega} (\nabla u^*)^2 ds \leq C \left(\|f\|_{D^o}^2 + \int_N g^2 ds \right)$$

In the following section it is proved that a change from Dirichlet to Neumann data on a single face is necessarily prohibited when the change takes place across the graph of a Lipschitz function. The strict convexity condition of (1.1) is also shown to be necessary. In the final section compact polyhedra are discussed for which the set N is necessarily *empty*.

2. PROOF OF THEOREM 1.1

The estimates that follow are scale invariant. Therefore to lighten the exposition a bit it will be assumed, when working near any vertex of the boundary of the compact polyhedron $\overline{\Omega}$, that the vertex is at least a distance of 4 units from any other vertex. Because $\partial\Omega$ is assumed to be a 2-manifold it will also be assumed that each edge that does not contain a given vertex v as an endpoint is at least 4 units from v and similarly each face. Consequently, by another application of the manifold condition, the picture that emerges is that the *truncated cones*

$$\mathcal{C}(v, r) = \{X \in \overline{\Omega} : |v - X| \leq r\}$$

for any vertex v and $0 \leq r \leq 4$ are homeomorphic to the closed ball \mathbb{B}^3 while the *cone bases*

$$\mathcal{B}(v, r) = \{X \in \overline{\Omega} : |v - X| = r\}$$

are homeomorphic to the closed disc \mathbb{B}^2 .

Define

$$\Omega_r = \Omega \setminus \bigcup_v \mathcal{C}(v, r), \quad 0 < r < 2$$

where the finite union is over all boundary vertices. Then each Ω_r is a Lipschitz domain (see, for example, §12.1 of [VV06] and Theorem 6.1 of [VV03] for a proof and generalizations in dimensions $n \geq 3$). Likewise the interiors of the *arches* defined by

$$(2.1) \quad \mathcal{A}(v, r, R) = \{X \in \overline{\Omega} : r \leq |v - X| \leq R\}, \quad 0 < r < R < 4$$

are Lipschitz domains.

In general neither of these kinds of domains have a uniform Lipschitz nature as $r \rightarrow 0$. Therefore the following *polyhedral Rellich identity* of [VV06] will be of use. It is proved as in [JK81] by an application of the Gauss divergence theorem, but with respect to the vector field

$$W := \frac{X}{|X|}, \quad X \in \mathbb{R}^3 \setminus \{0\}$$

when the origin is on the boundary of the domain.

Lemma 2.1. *Let \mathcal{A} be any arch (2.1) of the polyhedral domain $\Omega \subset \mathbb{R}^3$ and suppose u is harmonic in \mathcal{A} with $\nabla u^* \in L^2(\partial\mathcal{A})$. Then, taking the vertex v to be at the origin*

$$(2.2) \quad 2 \int_{\mathcal{A}} (W \cdot \nabla u)^2 \frac{dX}{|X|} = \int_{\partial\mathcal{A}} \nu \cdot W |\nabla u|^2 - 2 \partial_\nu u W \cdot \nabla u ds$$

Lemma 2.2. *With $\mathcal{A} = \mathcal{A}(v, r, R)$ and u as in Lemma 2.1*

$$(2.3) \quad 2 \int_{\mathcal{A}} (W \cdot \nabla u)^2 \frac{dX}{|X|} \leq \int_{\mathcal{B}(v, R)} |\nabla u|^2 ds + 2 \int_{\mathcal{B}(v, r)} (W \cdot \nabla u)^2 ds + 2 \int_{\partial\Omega \cap \mathcal{A}} |\partial_\nu u| |\nabla_t u| ds$$

Proof. The term $\nu \cdot W$ on the right of (2.2) is negative on $\mathcal{B}(v, r)$ and vanishes on $\partial\Omega$. Likewise the second integrand on the right of (2.2) is a perfect square on $\mathcal{B}(v, r)$, the negative of a square on $\mathcal{B}(v, R)$, and $W \cdot \nabla u$ is a tangential derivative on $\partial\Omega$. \square

The partition $D \cup N = \partial\Omega$ induces a decomposition of the Lipschitz boundaries $\partial\Omega_r$ into a Dirichlet part, a Neumann part, and bases $\mathcal{B}(v, r)$ of the cones removed from Ω . Define

$$N_r = (N \cap \partial\Omega_r) \bigcup_v \mathcal{B}(v, r)$$

and

$$D_r = \partial\Omega_r \setminus N_r.$$

This partition of $\partial\Omega_r$ satisfies the requirements of a creased domain in [Bro94]. See [VV06] pp. 586-587. (Including the bases in the Dirichlet part would also satisfy the requirements.) It will therefore be possible to invoke Brown's existence results in the domains Ω_r .

Similarly, *arches* $\mathcal{A} = \mathcal{A}(v, r, R)$ are creased Lipschitz domains with

$$N_r^R(v) = (N \cap \partial\mathcal{A}(v, r, R)) \cup \mathcal{B}(v, r) \cup \mathcal{B}(v, R)$$

and

$$D_r^R = \partial\mathcal{A} \setminus N_r^R$$

for each vertex v .

Brown's estimate from [Bro94] Theorem 2.1 is not scale invariant. However, the following special case is.

Theorem 2.3. (R. M. Brown) *Let $G \subset \mathbb{R}^n$ be a creased Lipschitz domain with $\partial G = D \cup N$. Then there exists a unique solution u to the mixed problem (1.2) for data f identically zero and $g \in L^2(N)$. Furthermore there is a constant C determined only by the scale invariant geometry of G, D and N and independent of g such that*

$$\int_{\partial G} (\nabla u^*)^2 ds \leq C \int_N g^2 ds$$

As is

Theorem 2.4. (R. M. Brown) *Let $G \subset \mathbb{R}^n$ be a creased Lipschitz domain with $\partial G = D \cup N$. Suppose that D is connected. Then there is a constant C such that for all harmonic functions u with $\nabla u^* \in L^2(\partial G)$*

$$\int_{\partial G} (\nabla u^*)^2 ds \leq C \left(\int_D |\nabla_t u|^2 ds + \int_N (\partial_\nu u)^2 ds \right)$$

Proof. Subtracting from u its mean value over D allows the Poincaré inequality over the connected set D . The conclusion still applies to u . \square

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^3$ be a compact polyhedral domain with 2-manifold boundary partitioned as $\partial\Omega = D \cup N$. Let v be a vertex and let j be a natural number. Suppose u is harmonic in the arch $\mathcal{A}(v, 2^{-j}, 2)$ with $\nabla u^* \in L^2(\partial\mathcal{A})$ and u vanishing on $D_{2^{-j}}^2$. Then there is a constant C independent of j so that*

$$\int_{\partial\Omega \cap \mathcal{A}(v, 2^{-j}, 2)} |\nabla u|^2 ds \leq C \left(\int_{\partial\Omega \cap N_{2^{-j}}} (\partial_\nu u)^2 ds + \int_{\mathcal{B}(v, 2^{-j})} (W \cdot \nabla u)^2 ds + \int_{\mathcal{A}(v, 1, 2)} |\nabla u|^2 dX \right)$$

Proof. For natural numbers $k \leq j$ and real numbers $1 \leq t \leq 2$ the arches $\mathcal{A}_{k,t} := \mathcal{A}(v, t2^{-k}, t2^{1-k})$ are geometrically similar Lipschitz domains. Therefore by the scale invariance of Brown's Theorem 2.3 above

$$\int_{\partial\Omega \cap \mathcal{A}_{k,t}} |\nabla u|^2 ds \leq C \int_{N_{t2^{1-k}}} (\partial_\nu u)^2 ds$$

with C independent of k . Take v to be the origin. For each k , integrating in $1 \leq t \leq 2$ and observing that $\nu = W$ or $-W$ on any cone base \mathcal{B}

$$\frac{1}{2} \int_{\partial\Omega \cap (\mathcal{A}_{k,1} \cup \mathcal{A}_{k,2})} |\nabla u|^2 ds \leq 2C \left(\int_{N \cap (\mathcal{A}_{k,1} \cup \mathcal{A}_{k,2})} (\partial_\nu u)^2 ds + \int_{\mathcal{A}_{k,1} \cup \mathcal{A}_{k,2}} (W \cdot \nabla u)^2 \frac{dX}{|X|} \right)$$

Summing on $k = 1, 2, \dots, j$ and using Lemma 2.2 on the arch $\mathcal{A}(v, 2^{-j}, R)$ for each $1 \leq R \leq 2$ together with the vanishing of u on $D_{2^{-j}}^2$ again

$$\begin{aligned} \frac{1}{2} \int_{\partial\Omega \cap \mathcal{A}(v, 2^{-j}, 2)} |\nabla u|^2 ds &\leq 4C \left(\int_{\partial\Omega \cap N_{2^{-j}}} (\partial_\nu u)^2 ds + \right. \\ &\quad \left. \int_{\mathcal{B}(v, R)} |\nabla u|^2 ds + 2 \int_{\mathcal{B}(v, 2^{-j})} (W \cdot \nabla u)^2 ds + 2 \int_{\partial\Omega \cap N_{2^{-j}}^R} |\partial_\nu u| |\nabla_t u| ds + \int_{\mathcal{A}(v, 1, 2)} |\nabla u|^2 dX \right) \end{aligned}$$

An application of Young's inequality ($2ab \leq \frac{1}{\epsilon} a^2 + \epsilon b^2$) allows the square of the tangential derivatives in the second to last term to be hidden on the left side and the normal derivatives to be incorporated in the first right side integral. Integrating in $1 \leq R \leq 2$ yields the final inequality. \square

By the same arguments, but using Theorem 2.4 and then Young's inequality in suitable ways for the D portion and the N portion of the last integral of Lemma 2.2, the next lemma is proved. For a given vertex, $D \cap \mathcal{C}(v, R_1)$ is connected if and only if any $D \cap \mathcal{A}(v, r, R_2)$ is connected.

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^3$ be a compact polyhedral domain with 2-manifold boundary partitioned as $\partial\Omega = D \cup N$. Let v be a vertex and let j be a natural number. Suppose $D \cap \mathcal{C}(v, 2)$ is connected and u is*

harmonic in the arch $\mathcal{A}(v, 2^{-j}, 2)$ with $\nabla u^* \in L^2(\partial\mathcal{A})$. Then there is a constant C independent of j so that

$$\int_{\partial\Omega \cap \mathcal{A}(v, 2^{-j}, 2)} |\nabla u|^2 ds \leq C \left(\int_{D_{2^{-j}}} |\nabla_t u|^2 ds + \int_{\partial\Omega \cap N_{2^{-j}}} |\partial_\nu u|^2 ds + \int_{\mathcal{B}(v, 2^{-j})} (W \cdot \nabla u)^2 ds + \int_{\mathcal{A}(v, 1, 2)} |\nabla u|^2 dX \right)$$

Let v be a vertex of the compact polyhedral domain Ω and consider the collection of nontangential approach regions $\Gamma(P)$ for $G = \Omega$ and parameter α (1.3) with $P \in \partial\Omega \cap \mathcal{C}(v, 4)$. By scale invariance each approach region can be *truncated* to a region

$$\Gamma^T(P) = \{X \in \Gamma(P) : |X - P| < (1 + \alpha) \text{dist}(X, \partial\mathcal{A}(v, r/2, 2r))\}, \quad |v - P| = r$$

so that the collections $\{\Gamma^T(P) : r \leq |v - P| \leq 2r\}$ can be extended in a uniform way to systems of nontangential approach regions *regular* in the sense of Dahlberg [Dah79] for the arches $\mathcal{A}(v, r/2, 4r)$.

Denote by w^T the nontangential maximal function of w with respect to the truncated cones Γ^T .

Denote the Hardy-Littlewood maximal operator on $\partial\Omega$ by \mathcal{M} . See, for example, [Ste70] pp.10-11 or [VV03] pp.501-502 for polyhedra.

For α large enough a geometric argument shows that there is a constant independent of P and w such that

$$(2.4) \quad w^*(P) \leq C\mathcal{M}(w^T)(P) + \max_K |w|, \quad P \in \partial\Omega \cap \mathcal{C}(v, 4)$$

where K is a compactly contained set in the Lipschitz domain Ω_2 .

Using Theorems 2.3 and 2.4 to estimate the truncated maximal functions introduces into the proofs of Lemmas 2.5 and 2.6 a doubling of the dyadic arches and therefore one dyadic term that is not immediately hidden by Young's inequality. Thus by the same proofs

Lemma 2.7. *With the same hypotheses as Lemma 2.5 there is a constant C independent of j so that*

$$\int_{\partial\Omega \cap \mathcal{A}(v, 2^{1-j}, 2)} (\nabla u^T)^2 ds - \frac{1}{2} \int_{\partial\Omega \cap \mathcal{A}(v, 2^{-j}, 2^{1-j})} |\nabla u|^2 ds \leq C \left(\int_{\partial\Omega \cap N_{2^{-j}}} (\partial_\nu u)^2 ds + \int_{\mathcal{B}(v, 2^{-j})} (W \cdot \nabla u)^2 ds + \int_{\Omega_1} |\nabla u|^2 dX \right)$$

Lemma 2.8. *With the same hypotheses as Lemma 2.6 there is a constant C independent of j so that*

$$\int_{\partial\Omega \cap \mathcal{A}(v, 2^{1-j}, 2)} (\nabla u^T)^2 ds - \frac{1}{2} \int_{\partial\Omega \cap \mathcal{A}(v, 2^{-j}, 2^{1-j})} |\nabla u|^2 ds \leq C \left(\int_{D_{2^{-j}}} |\nabla_t u|^2 ds + \int_{\partial\Omega \cap N_{2^{-j}}} |\partial_\nu u|^2 ds + \int_{\mathcal{B}(v, 2^{-j})} (W \cdot \nabla u)^2 ds + \int_{\Omega_1} |\nabla u|^2 dX \right)$$

Remark 2.9. Lemmas 2.5 and 2.6 apply to the negative terms of Lemmas 2.7 and 2.8. Consequently those terms may be removed from the inequalities.

2.1. The regularity problem. The regularity problem is the mixed problem for $\partial\Omega = D$.

Theorem 2.10. *Let $\Omega \subset \mathbb{R}^3$ be a compact polyhedral domain with 2-manifold connected boundary. Then for any $f \in W^{1,2}(\partial\Omega)$ the regularity problem is uniquely solvable and the estimate for the solution u*

$$\int_{\partial\Omega} |\nabla u^*|^2 ds \leq C \int_{\partial\Omega} |\nabla_t f|^2 ds$$

holds with C independent of f .

Proof. For each $\Omega_{2^{-j}}$ there is unique solution u_j to the mixed problem with $u_j = f$ on $D_{2^{-j}}$ and $\partial_\nu u_j = 0$ on $N_{2^{-j}}$ by Brown's existence result [Bro94]. By definition of the truncated approach regions in each vertex cone $\mathcal{C}(v, 4)$ the regions may be extended to a regular system of truncated approach regions for the $\partial\Omega \cap \partial\Omega_1$ part of the boundary. Thus the truncated nontangential maximal function can be defined there. By Lemma 2.8 and Remark 2.9, summing over all vertices,

using analogous estimates on the local Lipschitz boundary of $\partial\Omega$ outside of the vertex cones and using $W \cdot \nabla u_j = 0$ on the bases $\mathcal{B}(v, 2^{-j})$,

$$(2.5) \quad \int_{D_{2^{1-j}}} (\nabla u_j^T)^2 ds \leq C \left(\int_{D_{2^{-j}}} |\nabla_t f|^2 ds + \int_{\Omega_1} |\nabla u_j|^2 dX \right)$$

with C independent of j .

Subtracting from u_j the mean value m_f of f over $\partial\Omega$ does not change (2.5). Thus Poincaré (see [VV06]p.639 for polyhedral boundaries) can be applied over $\partial\Omega$ with constant independent of j in

$$(2.6) \quad \int_{\Omega_1} |\nabla u_j|^2 dX \leq \int_{D_{2^{-j}}} (u_j - m_f) \partial_\nu u_j ds = \int_{D_{2^{-j}}} (f - m_f) \partial_\nu u_j ds \leq C_\epsilon \int_{\partial\Omega} |\nabla_t f|^2 ds + \epsilon \int_{D_{2^{-j}}} |\nabla u_j|^2 ds$$

Applying Lemma 2.6 to the part of the integral over the regions $D_{2^{1-j}}^{2^{1-j}}$ and using $W \cdot \nabla u_j = 0$ again

$$\epsilon \int_{D_{2^{-j}}} |\nabla u_j|^2 ds \leq \epsilon C \left(\int_{D_{2^{-j}}} |\nabla_t f|^2 ds + \int_{\Omega_1} |\nabla u_j|^2 dX \right) + \epsilon \int_{D_{2^{1-j}}} |\nabla u_j|^2 ds$$

so that (2.6) yields

$$\frac{1}{2} \int_{\Omega_1} |\nabla u_j|^2 dX \leq (C_\epsilon + \epsilon C) \int_{\partial\Omega} |\nabla_t f|^2 ds + \epsilon \int_{D_{2^{1-j}}} |\nabla u_j|^2 ds$$

for all ϵ chosen small enough depending on C but not on j . Using this in (2.5) for ϵ chosen small enough gives

$$(2.7) \quad \int_{D_{2^{1-j}}} (\nabla u_j^T)^2 ds \leq C' \int_{\partial\Omega} |\nabla_t f|^2 ds$$

with the constant independent of j .

Given any compact subset of Ω , (2.7) together with $u_j = f$ on $D_{2^{-j}}$ for all j implies there exists a subsequence so that both u_{j_k} and ∇u_{j_k} converge uniformly on the compact set to a harmonic function u and its gradient respectively. A diagonalization argument gives pointwise convergence on all of Ω . Intersecting a compact subset K with the truncated approach regions yields compactly contained regions and corresponding maximal functions $\nabla u_{j_k}^{T,K} \rightarrow \nabla u^{T,K}$ uniformly. Thus by (2.7) and then monotone convergence, as Ω is exhausted by compact subsets K ,

$$(2.8) \quad \int_{\partial\Omega} (\nabla u^T)^2 ds \leq C \int_{\partial\Omega} |\nabla_t f|^2 ds$$

See [JK82] for these arguments.

A difficulty with the setup here is that the $\int_{D_{2^{1-j}}} (\nabla(u_j - u_k)^T)^2 ds$ for $k > j$ do not a priori have better bounds than the right side of (2.7). However, (2.7) together with weak convergence in $L^2(\partial\Omega_{2^{-j}})$ and pointwise convergence on the bases $\mathcal{B}(v, 2^{-j})$ shows that for each j and every $X \in \Omega_{2^{-j}}$ a subsequence of

$$u_k(X) = \int_{\partial\Omega_{2^{-j}}} \mathcal{P}_j^X u_k ds = \int_{D_{2^{-j}}} \mathcal{P}_j^X f ds + \sum_v \int_{\mathcal{B}(v, 2^{-j})} \mathcal{P}_j^X u_k ds$$

converges to $u(X)$, perforce with Poisson representation that must be an extension from $D_{2^{-j}}$ of f . Here \mathcal{P}_j^X is the Poisson kernel for the Lipschitz polyhedral domain $\Omega_{2^{-j}}$ and may be seen to be in $L^2(\partial\Omega_{2^{-j}})$ by Dahlberg [Dah77]. Consequently u has nontangential limits f on $\partial\Omega$, and by (2.4) and (2.11) the theorem is proved. \square

2.2. The mixed problem with vanishing Dirichlet data.

Theorem 2.11. *Let $\Omega \subset \mathbb{R}^3$ be a compact polyhedral domain with 2-manifold connected boundary. Then for any $g \in L^2(N)$ there is a unique solution u to the mixed problem (1.2) that vanishes on D and has Neumann data g on N . Further*

$$\int_{\partial\Omega} (\nabla u^*)^2 ds \leq C \int_N g^2 ds$$

Proof. Again by [Bro94] there exists a unique solution u_j in $\Omega_{2^{-j}}$ to the mixed problem so that $\partial_\nu u_j = g$ on $\partial\Omega \cap N_{2^{-j}}$, $W \cdot \nabla u_j = 0$ on the $\mathcal{B}(v, 2^{-j})$ and $u_j = 0$ on $D_{2^{-j}}$. Lemma 2.7 and Remark 2.9 imply

$$\int_{\partial\Omega \cap \partial\Omega_{2^{1-j}}} (\nabla u_j^T)^2 ds \leq C \left(\int_{\partial\Omega \cap N_{2^{-j}}} g^2 ds + \int_{\Omega_1} |\nabla u_j|^2 dX \right)$$

A Poincaré inequality independent of j is also needed here and is supplied by the following lemma. Polyhedral domains are naturally described as simplicial complexes. See for definitions and notations [Gla70] [RS72] [VV03] [VV06] or others.

Lemma 2.12. *Suppose u is harmonic in $\Omega_{2^{-j}}$ with $\partial_\nu u = g$ on $\partial\Omega \cap N_{2^{-j}}$, $\partial_\nu u = 0$ on the $\mathcal{B}(v, 2^{-j})$ and $u = 0$ on $D_{2^{-j}}$. Then*

$$\int_{\Omega_{2^{-j}}} |\nabla u|^2 dX \leq C \int_{\partial\Omega \cap N_{2^{-j}}} g^2 ds$$

with C independent of j .

Proof. By Green's first identity and Young's inequality

$$(2.9) \quad \int_{\Omega_{2^{-j}}} |\nabla u|^2 dX = \int_{\partial\Omega \cap N_{2^{-j}}} u \partial_\nu u ds \leq C_\epsilon \int_{\partial\Omega \cap N_{2^{-j}}} g^2 ds + \epsilon \int_{\partial\Omega \cap N_{2^{-j}}} u^2 ds$$

The polyhedron $\overline{\Omega}$ can be realized as a finite homogeneous simplicial 3-complex. A cone $\mathcal{C}(v, 1)$ is then the intersection of the ball $|X| \leq 1$ with the *star* $St(v, \overline{\Omega})$ in the 3-complex $\overline{\Omega}$ of the vertex v . Each 2-simplex σ^2 of $St(v, \overline{\Omega})$ that is also contained in N is contained in a unique 3-simplex $\sigma^3 \in St(v, \overline{\Omega})$. Let B denote the *unit* vector in the direction from the *barycenter* of σ^3 to v . Then $\sigma^2 \cap \{|X| \leq 1\}$ may be projected into the sphere $|X| = 1$ along lines parallel to B by $Q \mapsto Q + t_Q B$ onto a set contained in $\sigma^3 \cap \mathcal{B}(v, 1)$. The sets $\{Q + tB : Q \in \sigma^2 \cap N_{2^{-j}}^1(v) \text{ and } 0 \leq t \leq t_Q\}$ are contained in $\sigma^3 \cap \mathcal{A}(v, 2^{-j}, 1)$. Thus by the fundamental theorem of calculus for each $Q \in \partial\Omega \cap N_{2^{-j}}^1(v)$ and integrating $ds(Q)$

$$(2.10) \quad \int_{\partial\Omega \cap N_{2^{-j}}^1(v)} u^2 ds \leq C \left(\int_{\mathcal{A}(v, 2^{-j}, 1)} |\nabla u|^2 dX + \int_{\mathcal{B}(v, 1)} u^2 ds \right)$$

where the constant depends only on the projections, i.e. only on the finite geometric properties of the complex that realizes $\overline{\Omega}$ and not on j .

By the fundamental theorem, the connectedness of Ω_1 and the vanishing of u on the fixed nonempty set D

$$\int_{\partial\Omega_1} u^2 ds \leq C \int_{\Omega_1} |\nabla u|^2 dX$$

This together with (2.10) implies

$$\epsilon \int_{\partial\Omega \cap N_{2^{-j}}} u^2 ds \leq \epsilon C \int_{\Omega_{2^{-j}}} |\nabla u|^2 dX$$

and ϵ can be chosen independently of j so that (2.9) yields the lemma. \square

The lemma yields the analogue of (2.7)

$$(2.11) \quad \int_{\partial\Omega \cap \partial\Omega_{2^{1-j}}} (\nabla u_j^T)^2 ds \leq C \int_{\partial\Omega \cap N_{2^{-j}}} g^2 ds$$

Continuing to argue as in the proof of Theorem 2.10, this and the vanishing of the u_j on $D_{2^{-j}}$ produces a harmonic function u defined in Ω that is the pointwise limit of a subsequence of the u_j . In addition u satisfies

$$\int_{\partial\Omega} (\nabla u^T)^2 ds \leq C \int_N g^2 ds$$

which in turn yields the maximal estimate of the theorem.

To show that u assumes the correct data, (2.7) along with weak L^2 -convergence, pointwise convergence and the Poisson representation in each $\Omega_{2^{-j}}$ proves as before that u vanishes nontangentially on D . By constructing a Neumann function (possible by [JK81]) in analogy to the Green

function, or by using the invertibility of the classical layer potentials [Ver84], a Neumann representation of u in each Ω_{2-j} can be obtained so that $\frac{\partial u}{\partial \nu} = g$ nontangentially on N can be deduced by the same arguments.

Uniqueness follows from Green's first identity valid in polyhedra when $\nabla u^* \in L^2$. \square

2.3. Proof of Theorem 1.1. Recall the definition of the homogeneous Sobolev semi-norm (1.4).

Lemma 2.13. *When $\partial\Omega$ is connected and $\|f\|_{D^0}^2 = 0$, f is identically constant on D .*

The lemma says that f equals the same constant value on each component of D .

Proof. Because the semi-norm equals zero there is a sequence of extensions \tilde{f}_j of f and a sequence of numbers m_j so that by Poincaré in the second inequality

$$\int_D (f - m_j)^2 ds \leq \int_{\partial\Omega} (\tilde{f}_j - m_j)^2 ds \leq C \int_{\partial\Omega} |\nabla_t \tilde{f}_j|^2 ds \rightarrow 0$$

\square

Proof of Theorem 1.1. Choose an extension \tilde{f} of f so that $\int_{\partial\Omega} |\nabla_t \tilde{f}|^2 ds \leq 2\|f\|_{D^0}^2$. This is always possible by the lemma. Then from Theorem 2.10 there is a unique solution u_D with regularity data \tilde{f} and $\int_{\partial\Omega} (\nabla u_D^*)^2 ds \leq C\|f\|_{D^0}^2$. From Theorem 2.11 there is a unique solution u_N vanishing on D , with Neumann data $g - \partial_\nu u_D$ on N , and

$$\int_{\partial\Omega} (\nabla u_N^*)^2 ds \leq C \left(\int_N (\partial_\nu u_D)^2 ds + \int_N g^2 ds \right) \leq C \left(\|f\|_{D^0}^2 + \int_N g^2 ds \right)$$

The solution is $u = u_D + u_N$. Theorem 2.11 established uniqueness. \square

3. ON VIOLATIONS OF THE POSTULATES FOR THE PARTITION $\partial\Omega = D \cup N$

When D is empty the mixed problem is the Neumann problem and solvable for any data that has mean value zero on the boundary [Ver01]. We consider the two remaining postulates.

3.1. N is the union of a number (possibly zero) of closed faces of $\partial\Omega$. Solving the mixed problem means that every $W^{1,2}(D)$ function has a $W^{1,2}(\partial\Omega)$ extension. This observation raises the possibility that the mixed problem might be solvable when a given (open) face \mathcal{F} has nonempty intersection both with D and with N in such a way that $D \cap \mathcal{F}$ is an *extension domain*. Here we will only consider the possibility that this extension domain has a Lipschitz boundary [Ste70] and show that *the mixed problem is never solvable when this condition on the partition occurs*.

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz continuous function $y = \phi(x)$ with $\|\phi'\|_\infty \leq M$. Choose a point p_0 on the graph $(x, \phi(x))$ in the plane and consider the rectangle with width 2 parallel to the x -axis, length $8M$ and center p_0 . Locate the origin directly below p_0 and M units from the bottom of the rectangle. Here it will be convenient to name the region N that is *strictly below* the graph and contained in the rectangle. Call its complement in the rectangle D . Let (x, y, t) be the rectangular coordinates of \mathbb{R}^3 with *origin* coinciding with the origin of the plane. Let Z be the open right circular cylinder of \mathbb{R}^3 with center p_0 that intersects the plane in precisely the (open) rectangle.

The domain $\Omega = Z \setminus D \subset \mathbb{R}^3$ is *regular* for the Dirichlet problem. This follows by the Wiener test applied to each of the points of $\partial\Omega = \partial Z \cup D$ and the observation that the Newtonian capacity in \mathbb{R}^3 of a disc from the plane is proportional to its radius. See, for example, [Lan72] p. 165. Here the Lipschitz (or NTA) condition is also used. Consequently the Green function, $g = g^0$ for Ω with pole at the origin, is *continuous* in $\overline{\Omega} \setminus \{0\}$.

Approximating Lipschitz domains to Ω are constructed as follows. For each $\tau > 0$ define Lipschitz surfaces with boundary (the graph of ϕ) by

$$D_\tau = \{p + s(p - \tau e_3) : p \text{ is on the graph of } \phi \text{ and } 0 \leq s\} \cap Z$$

Here e_3 is the standard basis vector perpendicular to the xy -plane. Denote by H_τ the region of Z between D and D_τ above the graph of ϕ . Then the $\Omega_\tau = \Omega \setminus \overline{H}_\tau = Z \setminus \overline{H}_\tau$ are Lipschitz domains. Denote by g_τ the Green function for Ω_τ with pole at τe_3 .

Lemma 3.1. (i) $-\partial_t g(x, y, t)$ for $t > 0$ has continuous boundary values $\partial_\nu g := -\lim_{t \downarrow 0} g(x, y, t)/t$ at every point of D for which $y > \phi(x)$.

(ii) $\int_D (\partial_\nu g)^2 ds = +\infty$.

(iii) $\partial_t g(x, y, 0) = 0$ at every point of $N \setminus \{0\}$.

(iv) $\partial_\nu g \in L^2(\partial Z)$.

Proof. (i) follows by Schwarz reflection while (iii) follows by the symmetry in t of Ω and g . The maximum principle shows that the Green function for Z dominates from below the Green function for Ω , $g_Z \leq g \leq 0$. On ∂Z both Green functions vanish so that $\partial_\nu g_Z \geq \partial_\nu g \geq 0$ while $\partial_\nu g_Z$ is square integrable there, establishing (iv).

D. S. Jerison and C. E. Kenig's Rellich identity for harmonic measure ([JK82] Lemma 3.3) is valid on any Lipschitz domain G that contains the origin. It is

$$(n-2)w_G(0) = \int_{\partial G} (\partial_\nu g_G(Q))^2 \nu \cdot Q ds(Q)$$

with respect to the vector field X . Here $g_G(X) = F(X) + w_G(X)$ is the Green function for G , and F is the fundamental solution for Laplace's equation. Denote by w_τ , w and w_Z the corresponding harmonic functions for the Ω_τ , Ω and Z Green functions respectively.

By $Z \supset Z \setminus D = \Omega \supset \Omega_\tau$ and the maximum principle

$$\partial_\nu g_\tau \leq \partial_\nu g_Z \text{ on } \partial\Omega_\tau \setminus D \setminus D_\tau$$

$$0 < \partial_\nu g_\tau \leq g \text{ on } D$$

and

$$(3.1) \quad w_Z < w < w_\tau \text{ in } \Omega_\tau$$

For $Q \in D$ and $\nu = \nu_Q$ the outer unit normal to Ω_τ , $\nu \cdot (Q - \tau e_3) = \tau$, while for $Q \in D_\tau$, $\nu \cdot (Q - \tau e_3) = 0$. Formulating the Rellich identity with respect to the vector field $X - \tau e_3$ and using these facts ($n = 3$)

$$\begin{aligned} w_\tau(\tau e_3) &= \int_{\partial\Omega_\tau \setminus D \setminus D_\tau} (\partial_\nu g_\tau)^2 \nu \cdot (Q - \tau e_3) ds + \tau \int_D (\partial_\nu g_\tau)^2 ds \leq \\ &\quad \int_{\partial Z} (\partial_\nu g_Z)^2 \nu \cdot (Q - \tau e_3) ds + \tau \int_D (\partial_\nu g)^2 ds = w_Z(\tau e_3) + \tau \int_D (\partial_\nu g)^2 ds \end{aligned}$$

so that

$$\frac{w(\tau e_3) - w_Z(\tau e_3)}{\tau} < \frac{w_\tau(\tau e_3) - w_Z(\tau e_3)}{\tau} \leq \int_D (\partial_\nu g)^2 ds$$

and (ii) follows from (3.1) and $\tau \downarrow 0$. \square

For $\delta > 0$ define smooth subdomains of Ω

$$G_\delta = \{g < -\delta\}.$$

$\partial G_\delta \rightarrow \partial\Omega$ uniformly. The $\partial_\nu g|_{\partial G_\delta} ds$ are a collection of probability measures on \mathbb{R}^3 that have harmonic measure for Ω at the origin as weak-* limit.

By $G_\delta \uparrow \Omega$, Green's first identity, and monotone convergence

$$(3.2) \quad \int_{\Omega \setminus B_r} |\nabla g|^2 dX < \infty$$

for all balls centered at the origin.

With ϕ , N , D and Z as above define the half-cylinder domain $Z_+ = \{(x, y, t) \in Z : t > 0\}$. Then $D \cup N \subset \partial Z_+ \cap \{t = 0\}$.

Lemma 3.2. Suppose $\Delta u = 0$ in Z_+ , $\nabla u^* \in L^2(\partial Z_+)$, $\partial_\nu u \rightarrow^{n.t.} 0$ a.e. on N , and $u \rightarrow^{n.t.} 0$ a.e. on D . Let $Y \subset Z$ be a scaled cylinder centered at p_0 with $\text{dist}(\partial Y, \partial Z) > 0$. Let Y_+ be the corresponding half-cylinder. Then $u \in C(\overline{Y}_+)$.

Proof. The hypothesis on ∇u^* implies $u^* \in L^2(\partial Z_+)$ so that u and ∇u have nontangential limits a.e. on ∂Z_+ [Car62] [HW68]. Extend u to the bottom component of $Z \setminus D \setminus N$ by $u(x, y, t) = u(x, y, -t)$. By the vanishing of the Neumann data on N , $\Delta u = 0$ in the sense of distributions in the domain $\Omega = Z \setminus D$ and then classically.

Fix $d > 0$ and suppose $X \in \bar{Y}_+$ is of the form $X = (x, y, d)$ for $y \geq \phi(x) - Md$. Denote 3-balls of radius and distance to D comparable to d by B_d . Denote 2-discs in ∂Z_+ with radius comparable to d by Δ_d and let \oint denote integral average. Then by the mean value theorem, the fundamental theorem of calculus, the a.e. vanishing of u on D and the geometry of the nontangential approach regions

$$|u(X)| \leq \oint_{B_d(X)} |u| \leq Cd \left(\oint_{\Delta_d(x, y+2Md, 0)} \nabla u^* ds \right)$$

where C depends only on M . By absolute continuity of the surface integrals and $\nabla u^* \in L^2$ there is a function $\eta(d) \rightarrow 0$ as $d \rightarrow 0$ so that $\int_{\Delta_d} (\nabla u^*)^2 ds \leq \eta(d)$ for all $\Delta_d \subset \partial Z_+$. Consequently the Schwarz inequality now yields $|u(X)| \leq C\eta(d)$.

Suppose now X is of the form $X = (x, \phi(x) - Md, t)$ for $0 \leq t \leq d$. Because u has been extended

$$|u(X)| \leq \oint_{B_d(X)} |u| \leq \left(\oint_{B_d(x, \phi(x) - Md, d)} |u| \right) + d \left(\oint_{\Delta_d(x, \phi(x) - Md, 0)} \nabla u^* ds \right)$$

and $|u(X)| \leq 2C\eta(d)$. The lemma follows. \square

Partition ∂Z_+ by $N_+ = \bar{N}$, $D_+ = \partial Z_+ \setminus \bar{N}$ and $\partial Z_+ = N_+ \cup D_+$. For $3/4 > r > 0$ let Z^r be the scaled cylinder centered at p_0 of width $2r$ and length $8Mr$. Define the corresponding half-cylinders Z_+^r with

$$N_+^r = N_+ \cap \partial Z_+^r$$

(not a scaling of N_+) and

$$D_+^r = \partial Z_+^r \setminus N_+^r$$

With this partition Z_+^r is called a *split cylinder with Lipschitz crease*.

By (3.2) and the Fubini theorem, $g \in W^{1,2}(\partial Z_+^r \setminus \{t = 0\})$ for a.e. r .

Proposition 3.3. *Let g be the Green function for $\Omega = Z \setminus D$ with pole at the origin. For almost every $\frac{3}{4} > r > 0$ there exists no solution u to the L^2 -mixed problem (1.2) in the split cylinder with Lipschitz crease Z_+^r with boundary values $u \rightarrow^{n.t.} g \in W^{1,2}(D_+^r)$ and $\partial_\nu u \rightarrow \partial_\nu g = 0$ on N_+^r .*

Proof. Suppose instead that there is such a solution u with $\nabla u^* \in L^2(\partial Z_+^r)$. Then the first paragraph of the proof of Lemma 3.2 applies and, in particular, u extends to $Z^r \setminus D$ evenly and harmonically across N_+^r . The Dirichlet data that u takes a.e. on D_+^r is a continuous function, as is the Dirichlet data that u takes (continuously) on N_+^r . The Dirichlet data u takes a.e. on ∂Z_+^r will be shown to be a continuous function if it can be shown to be continuous across the boundary ∂N_+^r of the surface N_+^r . Lemma 3.2, scaled to apply to the split cylinders here, shows that the Dirichlet data is continuous across the Lipschitz crease part of ∂N_+^r . The same argument used there works on the other parts: Suppose $\text{dist}(X, \partial Z^r) = d$ for $X \in N_+^r$. Let $\Delta_d \subset \partial Z^r \cap D_+^r$ be a disc approximately a distance d from $X + de_3$. Then

$$|u(X) - \oint_{\Delta_d} g ds| \leq \left| \oint_{B_d(X)} u(Y) - u(Y + de_3) dY \right| + \left| \oint_{B_d(X+de_3)} u(Y) - \oint_{\Delta_d} g ds \right| \leq C\eta(d)$$

and the continuity across ∂N_+^r follows from the continuity of g and $\eta(d) \rightarrow 0$.

Thus the data u takes a.e. on ∂Z_+^r is a continuous function. Since also $u^* \in L^2(\partial Z_+^r)$ it follows that $u \in C(\bar{Z}_+^r)$. The evenly extended u is then continuous on \bar{Z}^r , harmonic in $Z^r \setminus D$ with the same Dirichlet data as g on $\partial(Z^r \setminus D)$. The maximum principle implies $u = g$.

Let g_r denote the Green function for $Z^r \setminus D$ with pole at a point $\{P\}$ of N_+^r . Again g_r is continuous in $\bar{Z}^r \setminus \{P\}$. Let $B \subset \bar{B} \subset Z^r \setminus D$ be a ball centered at P . Then by the maximum principle $cg \geq g_r$ on $\bar{Z}^r \setminus B$ for some constant c . By this domination, the vanishing of both g and g_r on $D_+^r \cap \{t = 0\}$ and (ii) of Lemma 3.1 applied to g_r , it follows that $\partial_\nu g$ which is not in $L^2(D)$ can

neither be square integrable over the smaller set $D_+^r \cap \{t = 0\}$. Since $u = g$ this contradicts the assumption on the nontangential maximal function of the gradient. \square

The nonsolvability of the L^2 -mixed problem in the split cylinders can be extended to nonsolvability in any polyhedron that has a Lipschitz graph crease on any face by a *globalization argument*. Let g and r be as in the Proposition. By using the approximating domains $Z^r \cap G_\delta$ as $\delta \rightarrow 0$, the Green's representation

$$g(X) = \int_{\partial Z^r} \partial_\nu F^X g - F^X \partial_\nu g ds - \int_{D \cap \overline{Z^r}} F^X d\mu^0, X \in Z^r \setminus D$$

can be justified where μ^0 is harmonic measure for $\Omega = Z \setminus D$ at the origin and F is the fundamental solution for Laplace's equation. Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function that is supported in a ball contained in Z^r centered at p_0 , and is identically 1 in a concentric ball B^r with smaller radius. Then define

$$u(X) = - \int_{D \cap \overline{Z^r}} F^X \chi d\mu^0$$

harmonic in \mathbb{R}^3 outside $\text{supp}(\chi) \cap D$. Similarly $g(X) - u(X)$ is harmonic inside B^r . Consequently $\nabla u^* \notin L^2(B^r \cap D)$ by applying a scaled (ii) of Lemma 3.1 to g again. Also

$$(3.3) \quad u(X) = - \int_{D \cap \overline{Z^r}} F^X(Q) (\chi(Q) - \chi(X)) d\mu^0(Q) - \chi(X) \int_{\partial Z^r} \partial_\nu F^X g - F^X \partial_\nu g ds + \chi(X) g(X)$$

The last term has bounded Neumann data on N and vanishing Dirichlet data on D . The Cauchy data of the middle term is smooth and compactly supported on $D \cup N$. For any $X \notin D$ the gradient of the first term is bounded by a constant, depending on χ , times

$$\left| \int_{D \cap \overline{Z^r}} F^X(Q) d\mu^0(Q) \right| \leq -F^X(0) + g^X(0) \leq \frac{1}{4\pi|X|}$$

Here g^X is the (negative) Green function for $\Omega = Z \setminus D$ with pole at X . Thus the first term is Lipschitz continuous on $D_+^r \cup N_+^r$. Altogether u has bounded Neumann data on N and Lipschitz continuous data on D while $\nabla u^* \notin L^2(D)$. Finally $\nabla u \in L_{loc}^2(\mathbb{R}^3)$ by (3.3) since this is true for χg .

Thus whenever a split cylinder Z_+^r can be contained in a polyhedral domain so that $\partial Z_+^r \cap \{t = 0\}$ is contained in a face and so that the Lipschitz crease is part of the boundary between the Dirichlet and Neumann parts of the polyhedral boundary, then the harmonic function u just constructed is defined in the entire polyhedra domain. Its properties suffice to compare it with any solution w in the class $\nabla w^* \in L^2$ by Green's first identity $\int |\nabla u - \nabla w|^2 dX = \int (u - w) \partial_\nu (u - w) ds$. Regardless of the nature of the partition away from Z_+^r , when w has the same data as does u it must be concluded, as in Proposition 3.3, that it is identical to u . This establishes

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^3$ be a compact polyhedral domain with partition $\partial\Omega = D \cup N$. Let \mathcal{F} be an open face of $\partial\Omega$ such that $\mathcal{F} \cap D$ is a Lipschitz domain of \mathcal{F} with nonempty complement $\mathcal{F} \cap N$. Then there exist mixed data (1.2) for which there are no solutions u in the class $\nabla u^* \in L^2(\partial\Omega)$.*

3.2. Whenever a face of N and a face of D share a 1-dimensional edge as boundary, the dihedral angle measured in Ω between the two faces is less than π . Continue to denote points of \mathbb{R}^3 by $X = (x, y, t)$. Define D to be the upper half-plane of the xy -plane. Introduce polar coordinates $y = r \cos \theta$ and $t = r \sin \theta$, let $\pi \leq \alpha < 2\pi$ and define N to be the half-plane $\theta = \alpha$. The crease is now the x -axis.

Define

$$b(X) = r^{\frac{\pi}{2\alpha}} \sin\left(\frac{\pi}{2\alpha}\theta\right)$$

for X above $D \cup N$. These are Brown's counterexample solutions for nonconvex plane sectors [Bro94]. The Dirichlet data vanishes on D while the Neumann vanishes on N , and $\nabla b^* \notin L^2$.

These solutions are globalized to a compact polyhedral domain with interior dihedral angle α :

Denote by Θ the intersection of a (large) ball centered at the origin and the domain above $D \cup N$. Then $b(X)$ is represented in Θ by

$$b(X) = \int_{\partial\Theta \setminus D} \partial_\nu F^X b ds - \int_{\partial\Theta \setminus N} F^X \partial_\nu b ds$$

Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function as before, but centered at the *origin* on the crease. Define

$$u(X) = \int_N \partial_\nu F^X \chi b ds - \int_D F^X \chi \partial_\nu b ds$$

As before, u is harmonic everywhere outside $\text{supp}(\chi) \cap (D \cup N)$ and $\nabla u^* \notin L^2(\text{supp}(\chi) \cap (D \cup N))$. Also

$$(3.4) \quad u(X) = \int_{N \cap \bar{\Theta}} \partial_\nu F^X(Q) (\chi(Q) - \chi(X)) b(Q) ds(Q) \\ - \int_{D \cap \bar{\Theta}} F^X(Q) (\chi(Q) - \chi(X)) \partial_\nu b(Q) ds(Q) - \chi(X) \int_{\partial \Theta \setminus N \setminus D} \partial_\nu F^X b - F^X \partial_\nu b ds + \chi(X) b(X)$$

Again the boundary values around the support of χ are the issue. The last two terms are described just as the middle and last after (3.3). The *gradient* of the second term is *bounded* because the integral over D can be no worse than, for example, $\int_0^1 dx \int_0^1 \frac{1}{\sqrt{x^2+r^2}} \frac{dr}{r^\beta} < \infty$ for any $\beta < 1$ (e.g. $\beta = 1 - \frac{\pi}{2\alpha}$).

For a $\frac{\partial}{\partial X_j}$ derivative define tangential derivatives (in Q) to any surface with unit normal ν by $\partial_i^t = \nu_i \partial_j - \nu_j \partial_i$. Then by the harmonicity of F away from X and the divergence theorem in Θ , the $\frac{\partial}{\partial X_j}$ derivative of the first integral equals the sum in i of

$$\int_{N \cap \bar{\Theta}} \partial_i F^X \partial_i^t ((\chi - \chi(X)) b) ds$$

plus integrals over $\partial \Theta \setminus N \setminus D$ (b vanishes on D) that will all be *bounded* since X is near the support of χ . When the tangential derivative falls on b the integral is bounded like the second term of (3.4). The remaining integral has boundary values in every L^p for $p < \infty$ by singular integral theory. (In fact, it too is bounded by a closer analysis, thus making it consistent with the example from Section 3.1.)

Finally $\nabla u \in L_{loc}^2(\mathbb{R}^3)$ by its now established properties and the corresponding property for b . The argument using Green's first identity as at the end of Section 3.1 is justified and

The solutions u can now be placed in polyhedral domains that have interior dihedral angles greater than or equal to π and provide mixed data for which no L^2 -solution can exist.

4. POLYHEDRAL DOMAINS THAT ADMIT ONLY THE TRIVIAL MIXED PROBLEM

Consider the L^2 -mixed problem for the unbounded domain exterior to a compact polyhedron. When the polyhedron is convex the requirement of postulate (iii) of (1.1) eliminates all but the trivial partition from the class of well posed mixed problems. In this case we will say that the exterior problem is *monochromatic*.

The mixed problem for a compact polyhedral domain can also be monochromatic for the *interior* problem. An example is provided by the regular compound polyhedron that is the union of 5 equal regular tetrahedra with a common center, a picture of which may be found as Number 6 on Plate III between pp.48-49 of H. S. M. Coxeter's book [Cox63]. An elementary arrangement of plane surfaces that elucidates the *local* element of this phenomenon is found upon considering the domain of \mathbb{R}^3 that is the union of the upper half-space together with all points (x, y, t) with (x, y) in the first quadrant of the plane, i.e. the union of a half-space and an infinite wedge. The boundary consists of 3 faces: the 4th quadrants of both the xt and yt -planes and the piece of the xy -plane outside of the 1st quadrant of the xy -plane. The requirement of postulate (iii) is met only by the *negative* t -axis. But no color change is possible there because any color on either of the 4th quadrants must be continued across the *positive* x or y -axis to the 3rd face of the boundary. On the other hand, a color change is possible for the complementary domain and is possible for the exterior domain to the compound of 5 tetrahedra.

Is there a polyhedral surface with a finite number of faces for which both interior and exterior mixed problems are monochromatic?

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